Lecture 5. ***Classification of points of stationary (special points) on the phase plane***

Let's consider phase portraits of a linear dynamical system of the 2nd order.

Let a linear dynamical system of the *2nd* order ($u(t) \equiv 0$) is given:

$$\begin{cases} \dot{x}_1 = a_{11}x_1 + a_{12}x_2 \\ \dot{x}_2 = a_{21}x_1 + a_{22}x_2 \end{cases} \qquad (6.5)$$

or in matrix form:

$$\dot{X} = AX.$$

Let's consider a stationary point, in which variables change speed is equal to zero

$\dot{x} = 0$, so $AX = 0$, but $\det A \neq 0$, that is therefore $X \equiv 0$, so

$$X = \begin{vmatrix} x_1 \\ x_2 \end{vmatrix} = \begin{vmatrix} 0 \\ 0 \end{vmatrix}.$$

Hence, coordinates of rest points are equal to $\begin{Bmatrix} x_1 = 0 \\ x_2 = 0 \end{Bmatrix}$.

So we can make a conclusion: *the position of balance for a linear system is the beginning of coordinates.*

Characteristic equation of the system (6.5) has the form:

$$det(A\text{-}sI) = s^2 + a_2 s + a_1 = 0.$$

Let us consider some more general types of phase trajectories and special points depending on the type and location of the roots of characteristic equation of the system (6.5).

I. Let a characteristic equation have real roots of the system

1) Let the roots of characteristic equation of the system be real and negative (fig. 6.11a):

$S_1 = a_1;\ S_2 = a_2;\quad a_1 < 0,\ a_2 < 0,\ a_1, a_2$ - const.

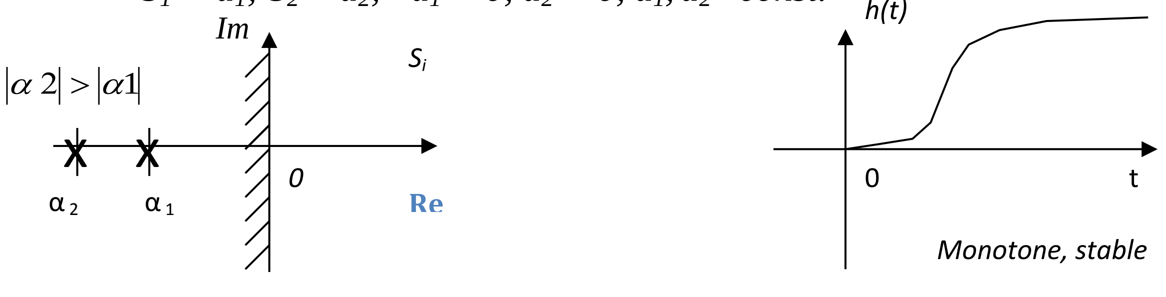


Fig.6.11a. Roots location and corresponding transient process

In this case the phase portrait corresponds to a special point, *stable knot* (fig.6.12). Here the straight line is a degenerated trajectory $e^{-\alpha t}$.

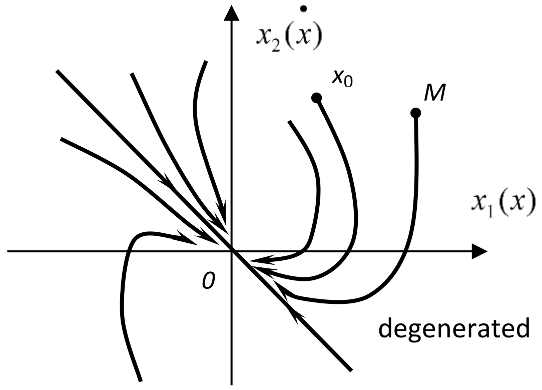


Fig.6.12.A stable knot

1) Let roots of characteristic equation of the system be real, positive (fig. 6.13a) :
   $S_1 = a_1$; $S_2 = a_2$; $a_1 > 0, a_2 > 0, a_1, a_2$ *- const.*

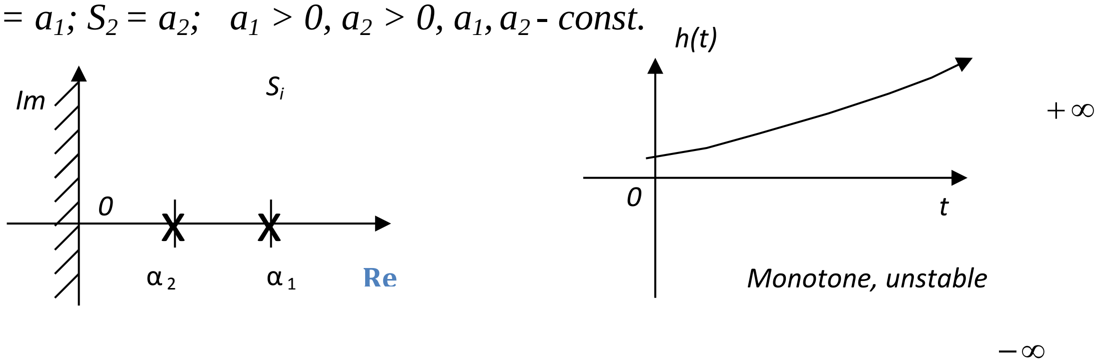


Fig.6.13a. Roots location and the corresponding transient process

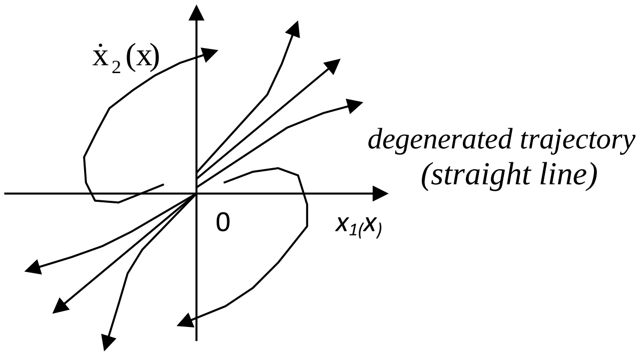


Fig.6.13b. An unstable knot

In this case the phase portrait corresponds to a special point, *unstable knot* (fig.6.13b). Balance point is instable; here the straight line is a degenerated trajectory $e^{\alpha t}$.

3) Let the roots of characteristic equation of the system be real, they have different signs (fig. 6.14a):

$$S_1 = a_1;\ S_2 = a_2;\quad a_1 > 0,\ a_2 < 0,\ a_1, a_2 \text{ - } const.$$

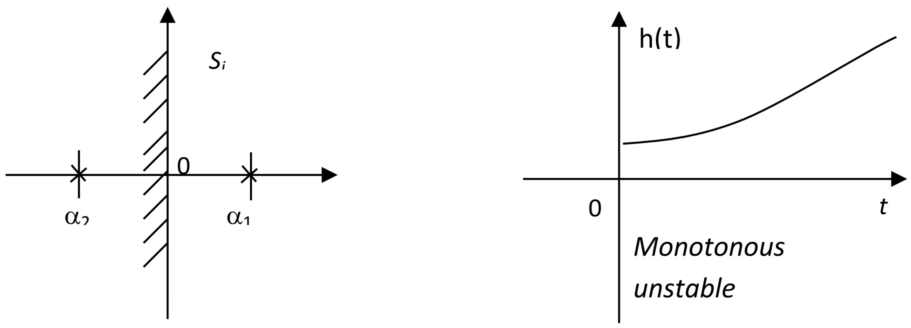


Fig.6.14a. The location of real roots of different signs and the corresponding transient process

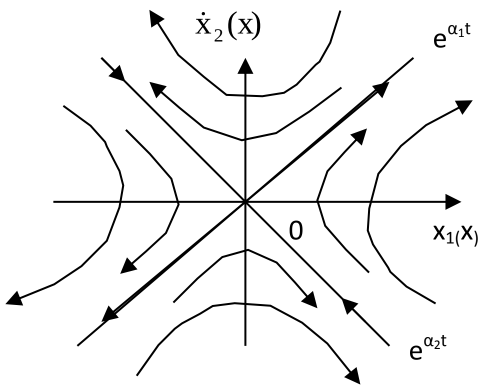


Fig.6.14b. A special point like a saddle

In this case the phase portrait corresponds to *a special point like a saddle* (always instable) (fig.6.14b); $e^{\alpha 1t}$, $e^{\alpha 2t}$ are degenerated trajectories.

II. *Let the roots of characteristic equation of the system be complex conjugate:* $s_{1,2} = +\alpha \pm j\beta$.

1) The real part of the roots is negative, so α < 0 (fig. 6.15a).

The system is stable, because the real part of the roots is negative: the phase trajectories are of spiral type, special point is stable focal point (fig.6.15b).

Fig.6.15a. Location of complex conjugate roots and the corresponding transient process

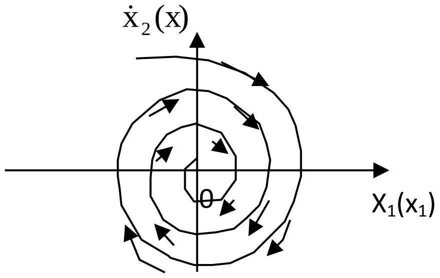


Fig.6.15b. A stable focal point

2) The real part of the roots is positive, so $a > 0$ (fig. 6.16a).

In this case the phase trajectories are of spiral type: special point is *instable focal point* (Fig.6.16b).

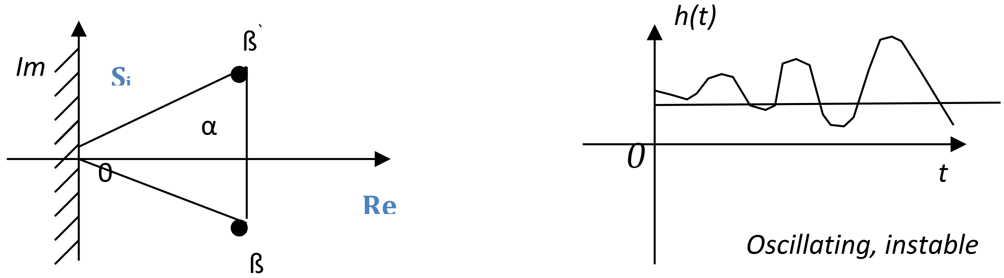


Fig.6.16a. Location of complex conjugate roots and the corresponding transient process

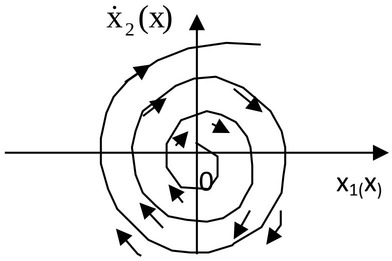
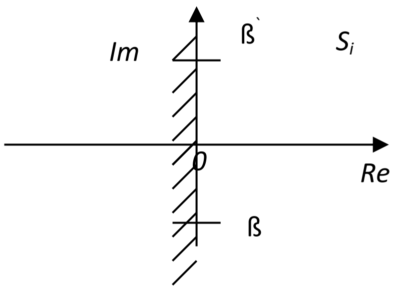


Fig.6.16b. An instable focal point

III. Let the roots of characteristic equation of the system be purely imaginary: $S_{1,2} = \pm j\beta$ (fig. 6.17a)



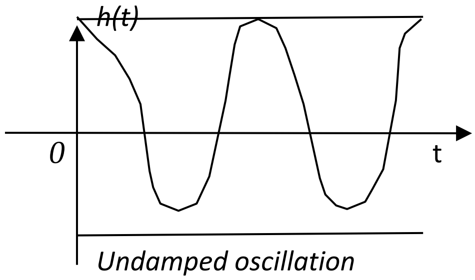


Fig.6.17a. Location of imaginary roots and
the corresponding transient process

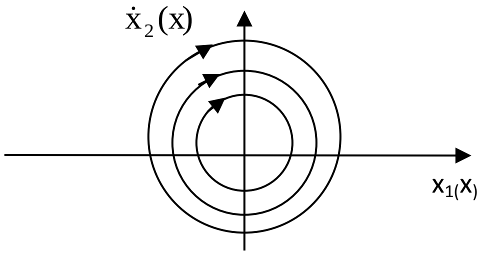


Fig.6.17b. A special point of center type

In this case the phase portrait corresponds to *a special point of center type* (fig.6.17b). We will mention once more, that motion stopping corresponds to special or balanced points of the system.

Phase portraits of linear systems allow to present visually set of all possible system motions on phase plane. Using of phase portraits appeared to be very convenient for construction of systems with variable structure (*SVS*).